MATH2050C Selected Solution to Assignment 11

Section 5.3

(3) Define a sequence $\{x_n\}$ be $|f(x_{n+1})| \le |f(x_n)|/2$ where $x_1 \in [a,b]$ is arbitrary. We have $|f(x_n)| \le |f(x_1)|/2^{n-1}$, and so $\lim_{n\to\infty} |f(x_n)| \le \lim_{n\to\infty} |f(x_1)|2^{1-n} = 0$. By Bolzano-Weierstrass, there is a subsequence $\{x_{n_j}\}$ converging to some $z \in [a,b]$. By continuity, $f(z) = \lim_{j\to\infty} f(x_{n_j}) = 0$. (Note that $\{a_n\}$ tends to 0 if and only if $\{|a_n|\}$ tends to 0.)

(5) $p(-10) = 2991, p(0) = -9$, and $p(2) = 63$. By the theorem on Existence of Zeros, there is a zero in $(-19, 0)$ and another in $(0, 63)$.

(6) The function g satisfies $g(0) = f(0) - f(1/2)$ and $g(1/2) = f(1/2) - f(1) = f(1/2) - f(0) = -g(0)$. It is also continuous on $[0, 1/2]$. If $g(0) = 0$, we are done. If $g(0) \ne 0, g(0)g(1/2) = -g(0)^2 < 0$, so the desired conclusion comes from the theorem on Existence of Zeros.

Note. Borsuk-Ulam Theorem asserts that any continuous mapping F from the unit sphere

$$S = \left\{ x \in \mathbb{R}^n : \ x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \right\},$$

to $\mathbb{R}^n$ satisfies the following property: There exists a point $p \in S$ so that $F(p) = F(-p)$. This exercise is essentially the case $n = 1$.

(12) The function $g(x) = \cos x - x^2$ satisfies $g(0) = 1 > 0$ and $g(\pi/2) < 0$, so there is some $x_0 \in (0, \pi/2)$ such that $g(x_0) = 0$. Since $\cos x$ is strictly decreasing and x^2 is strictly increasing on $[0, \pi/2]$, g is strictly decreasing and x_0 is the unique zero for g. It means $g(x) > 0$, that is, $\cos x > x^2$ on $[0, x_0)$ and $g(x) < 0$, that is, $\cos x < x^2$ on $(x_0, \pi/2]$. It implies $f(x) = \cos x$ on $[0, x_0)$ and $f(x) = x^2$ on $(x_0, \pi/2]$. The conclusion comes from the fact that $\cos x > \cos x_0$ on $[0, x_0)$ and $x^2 > x_0^2$ on $(x_0, \pi/2]$.

Supplementary Exercise

1. Let f be continuous on $[a,b]$. For $x_1, x_2, \cdots, x_n \in [a,b]$, show that there is some $\xi \in [a,b]$ such that
$$f(\xi) = \frac{1}{n}(f(x_1) + f(x_2) + \cdots + f(x_n)).$$
Is the conclusion still valid when $[a,b]$ is replaced by (a,b)?

   Solution. The continuity of f on $[a,b]$ implies that the range of $[a,b]$ under f is the interval $[m, M]$ where m and M are respectively the minimum point and maximum point of f. Now, $m \le f(x_j) \le M$ for $j = 1, \cdots, n$ implies

$$m \le \frac{1}{n}(f(x_1) + \cdots + f(x_n)) \le M,$$

   so the point $((f(x_1) + \cdots + f(x_n))/n$ belongs to $[m, M]$, the range of $[a,b]$ under f. The desired conclusion follows.

   The conclusion still holds when $[a,b]$ is replaced by (a,b). Just consider the interval $[x_1, x_n]$ where $x_1 \le x_2 \le \cdots \le x_n$.

2. Let h be an increasing function on some interval (a, b), that is, $h(x) \leq h(y)$ for $x \leq y$.
   (a) Show that $\lim_{x \to x_0^+} h(x)$ and $\lim_{x \to x_0^-} h(x)$ always exist for every $x_0 \in (a, b)$.
   (b) Show that h is continuous on $[a, b]$ if and only if the range of h is $[h(a), h(b)]$.
   (c) Optional. Show that if for a given number $k > 0$, the set $\{z \in (a, b) : \lim_{x \to z^+} h(x) - \lim_{x \to z^-} h(x) \geq k\}$ is a finite set.
   (d) Optional. Deduce from (c) that h has at most countably many points of discontinuity.

Solution. (a) Claim $\alpha \equiv \sup\{h(x) : x \in [a, x_0)\}$ is the left hand limit and $\inf\{h(x) : x \in (x_0, b]\}$ is the right hand limit. Since h is increasing, we have $h(x) \leq h(b)$ which means α is a finite number. To prove it is the left hand limit of h at x_0, we need to show, for $\varepsilon > 0$, there is some δ such that $|h(x) - \alpha| < \varepsilon$ for $x \in (x_0 - \delta, x_0)$. By the definition of α, for $\varepsilon > 0$, there is some $h(x_1), x_1 < x_0$, such that $h(x_1) + \varepsilon > \alpha$. By monotonicity, it follows that $h(x) + \varepsilon \geq h(x_1) + \varepsilon > \alpha$ for all $x, x \in [x_1, x_0)$, so $|h(x) - \alpha| < \varepsilon$, done. The right hand limit can be treated in a similar manner.

(b) When h is continuous on $[a, b]$, its range is equal to $[m, M]$ where m and M are respectively the minimum and maximum of h. As h is increasing, $[m, M]$ is equal to $[h(a), h(b)]$. On the other hand, if h is not continuous at some $x_0 \in (a, b)$, $\lim_{x \to x_0^-} h(x) < \lim_{x \to x_0^+} h(x)$ according to (a). Then any point k satisfying $\lim_{x \to x_0^-} h(x) < k < \lim_{x \to x_0^+} h(x)$ lies outside of the range of h, hence $[h(a), h(b)]$ cannot be an interval. The case of possible discontinuity at a or b can be treated similarly.

(c) Suppose there are N many points in this set. By monotonicity,

$$Nk \leq \sum_{n=1}^{N} \left(\lim_{x \to x_j^+} h(x) - \lim_{x \to x_j^-} h(x) \right) \leq h(b) - h(a),$$

which imposes the following bound on N:

$$N \leq \frac{h(b) - h(a)}{k} \ .$$

Hence this set is finite for each given k.

(d) Let $E_n = \{z \in (a, b) : \lim_{x \to z^+} h(x) - \lim_{x \to z^-} h(x) \geq 1/n\}$. By (a), any discontinuity of h belongs to some E_n. Therefore, the discontinuity set which is equal to $\cup_{n=1}^{\infty} E_n$ is a countable set.

Note. We will discuss this problem in class.

Existence of Zeros for Continuous Functions

We give a proof of the theorem on the existence of zeros different from the bisection method. Proposition 1 will be used many times in this course and 2060.

Proposition 1. Let f be defined on (a,b) and continuous at $x_0 \in (a,b)$. When $\alpha = f(x_0) > 0$, there is some small $\delta > 0$ such that $f(x) > \alpha/2$ for all $x \in (x_0-\delta, x_0+\delta)$. When $f(x_0) < 0$, there is some small $\delta > 0$ such that $f(x) < f(x_0)/2$ for all $x \in (x_0-\delta, x_0+\delta)$.

Proof. Let $\varepsilon = \alpha/2$. There exists some $\delta > 0$ such that $|f(x) - f(x_0)| = |f(x) - \alpha| < \alpha/2$ for all $x \in (x_0-\delta, x_0+\delta)$. (We could choose δ so small that this interval is included in (a,b).) It follows that $f(x) - \alpha > -\alpha/2$, that is, $f(x) > \alpha/2$, on this interval.

Here are some remarks.

- First, by choosing a smaller δ, $(x_0-\delta, x_0+\delta)$ could be replaced by $[x_0-\delta, x_0+\delta]$. Second, it means in particular that $f > 0$ on $(x_0-\delta, x_0+\delta)$.
- When f is right hand continuous at $x_0 \in [a,b)$ or left hand continuous, the conclusion holds on $[x_0, x_0+\delta]$ or $[x_0-\delta, x_0]$ repectively.
- Similar results hold when $f(x_0) < 0$. Simply consider $-f$.

Theorem 2. Let f be continuous on $[a,b]$ satisfying $f(a)f(b) < 0$. There exists some $c \in (a,b)$ such that $f(c) = 0$.

Proof. Without loss of generality assume $f(a) < 0$ and $f(b) > 0$. Consider the set $E = \{c \in [a,b] : \ f > 0 \text{ on } [a,c]\}$. By Proposition 1, f is positive on $[a, a+\delta]$ for some small δ. Hence E is nonempty by taking $c = a+\delta$. On the other hand, E is bounded above by b. By Order-Completeness Property, $\xi = \sup E \le b$ exists. By the definition of supremum, we can find a sequence $z_n \in E$ such that $\lim_{n\to\infty} z_n = \xi$. By continuity, $0 \ge \lim_{n\to\infty} f(z_n) = f(\xi)$. On the other hand, if $f(\xi) < 0$, Proposition 1 asserts that $f(x) < 0$ for $x \in [\xi-\delta_1, \xi+\delta_1]$ for some small δ_1, thus $f < 0$ on $[a, \xi+\delta_1]$, contradicting the fact that ξ is the supremum of E. One must have $f(\xi) = 0$.